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Error estimates for $\pi\pi$ scattering threshold parameters in Chiral Perturbation Theory to two loops.[1]

J. Nieves[2] and E. Ruiz Arriola[3]

Departamento de Física Moderna, Universidad de Granada, E-18071 Granada, Spain.
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Abstract

Using the analysis of ChPT to two loops, we make an error analysis of the low energy parameters, based on the errors for the one loop low energy parameters and the resonance saturation mechanism. Thus, the predictive power of the effective field theory is quantified on the basis of the experimental uncertainties.

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Chiral perturbation theory (ChPT)is the effective pion field theory where the expansion parameter is m^2/f^2, with m the pion mass and f the weak pion decay constant. At some finite order an increasing number of undetermined parameters is generated which can, so far, only be fixed by experimental data. Besides the theoretical uncertainty introduced by the finite number of terms considered in the expansion, there is an additional experimentally induced uncertainty. The low energy parameters inherit such an uncertainty, thus limiting in practice the predictive power of the effective field theory; if the errors at a given order, say $\mathcal{O}(m^{2n}/f^{2n})$ are larger than the contributions of the next order $\mathcal{O}(m^{2n+2}/f^{2n+2})$, the calculation of the latter becomes useless, unless more accurate experiments are performed.

To make our points quantitative, we consider $\pi\pi$ scattering as a prototype reaction to study the role played by chiral symmetry breaking and the validity of the chiral expansion within QCD at low energies. The scattering lengths and effective ranges for the isospin $I = 0, 1, 2$ have been computed in SU(2) ChPT at tree level [1], one loop [2] and two loops [3, 4], in a power series expansion

$$a_{IJ} = a_{IJ}^{\rm tree} + a_{IJ}^{1\,\rm loop} + a_{IJ}^{2\,\rm loop} + \cdots \quad (1)$$

$$b_{IJ} = b_{IJ}^{\rm tree} + b_{IJ}^{1\,\rm loop} + b_{IJ}^{2\,\rm loop} + \cdots \ . \quad (2)$$

An error analysis from the point of view of the predictive power of ChPT is missing in Refs. [3, 4], and it is the main subject of the present work. At tree level, the number of parameters involved are two: the pion weak decay constant $f_\pi = 93.2$MeV and the pion mass $m_\pi = 139.6$MeV. For the purposes of the present discussion the experimental error bars in these parameters can effectively be taken to be zero. At the one loop level one has four, in principle undetermined, parameters $\bar{l}_1, \bar{l}_2, \bar{l}_3, \bar{l}_4$. These parameters can be fixed by computing several processes, and we will take their central values to be the same as in Ref. [4] (there called sets **I** and **II**). In the parameter Set **I** $\bar{l}_1$ and $\bar{l}_2$ are obtained from the absolute values of the K_{l4} form factors using a dispersive one loop calculation for three flavours [5], and $\bar{l}_3$ and $\bar{l}_4$ are fixed as in Ref. [2]. This yields

$$\bar{l}_1 = -1.7 \pm 1.0\,, \quad \bar{l}_2 = 6.1 \pm 0.5\,, \quad \bar{l}_3 = 2.9 \pm 2.4\,, \quad \bar{l}_4 = 4.3 \pm 0.9 \qquad \text{Set } \mathbf{I} \quad (3)$$

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[2]e-mail: jmnieves@ugr.es
[3]e-mail:earriola@ugr.es

In what follows we will take these four parameters to be statistically uncorrelated. At the one loop level the whole SU(2) $\pi\pi$ amplitude information can be gathered into six coefficients, which in Ref. [3, 4] have been called $b_1, \ldots, b_6$. Ideally, these parameters could be extracted from a low energy analysis of $\pi\pi$ scattering experimental data. The data are, however, too poor and thus the authors of Ref. [4] preferred to make use of a resonance saturation hypothesis, which is believed to work at scales $\mu \sim .5 - 1.\text{GeV}$, together with the values for the $\bar{l}$'s deduced from several sources and quoted above. The parameters can be expressed (see Eq. D-1 of Ref. [4]) in terms of $\bar{l}_{1,2,3,4}$ and some resonance contributions (see Eq.(5.7) of Ref. [4]). We implement the scale ambiguity in the determination of the $b's$ by assuming an error on it, it is to say we take

$$\mu = 750 \pm 250\,\text{MeV}\,. \tag{4}$$

The method for Ref. [4] allows to define another parameter set. If the D-wave $\pi\pi$ scattering lengths for isospin $I = 0$ and $I = 2$, a_{02} and a_{22} respectively, are fixed at the two loop level one gets parameter Set **II**,

$$\bar{l}_1 = -0.8 \pm 4.8\,, \quad \bar{l}_2 = 4.5 \pm 1.1\,, \quad \bar{l}_3 = 2.9 \pm 2.4\,, \quad \bar{l}_4 = 4.3 \pm 0.9 \qquad \text{Set } \mathbf{II} \tag{5}$$

To obtain $\bar{l}_1$ and $\bar{l}_2$ and their errors, we have propagated the errors in μ, a_{02}, a_{22}, $\bar{l}_3$ and $\bar{l}_4$ in the formula for the D-wave scattering length given in Ref. [4]. This procedure generates a correlation between $\bar{l}_1$ and $\bar{l}_2$ which has to be taken into account when calculating errors in quantities depending on the previous parameters. Note that our central values for $\bar{l}_1$ and $\bar{l}_2$ numbers are not exactly theirs since they take $\mu = 1\text{GeV}$. Furthermore, in Ref. [4] no error estimates are quoted for $\bar{l}_1$ and $\bar{l}_2$. If we had taken $\mu = 1.00 \pm 0.25\text{GeV}$, we would have obtained $\bar{l}_1 = -1.5 \pm 5.8$ and $\bar{l}_2 = 4.5 \pm 1.1$, in agreement with their quoted central value.

For the parameter Set **I**, the errors in the $\bar{l}_{1,2,3,4}$ obviously produce an uncertainty both in the one loop and the two loop corrections to the scattering lengths and effective ranges. Thus, the error in the two loop contribution is correlated with that in the one loop contribution. The uncertainty in the scale and in the resonance contributions, affect the error of the two loop corrections only. If the parameters $\bar{l}_1$ and $\bar{l}_2$ are fixed, as in Set **II**, to a given observable with two loop accuracy, there also appears a strong correlation between the one and the two loop contributions, in addition to the obvious $\bar{l}_1$-$\bar{l}_2$ correlation. Thus, the error of the sum cannot be obtained by simply adding the errors in quadrature of one and two loop contributions.

In any case, for the two loop calculation to be numerically meaningful these one loop uncertainties have to be significantly smaller than the corrections due to the two loop calculation. At the same time the two loop correction has to be significantly smaller than the one loop correction, for a convergent expansion. Let $a^{(n)}$ and $\Delta a^{(n)}$ be the n-loop central value and error of an observable. Thus, to be *predictive* and *convergent* at the n-loop level one ought to have the relation,

$$\Delta a^{(n)} << a^{(n+1)} << a^{(n)}. \tag{6}$$

The easiest way to propagate errors, with or without correlations is via a Monte Carlo method. We generate a sufficiently large sample of low energy parameters $\bar{l}_{1,2,3,4}$, resonance contributions[4] and scales μ randomly distributed according to a gaussian in the case of parameter Set **I**. For parameter Set **II**, samples of the D-wave scattering lengths, a_{02}, a_{22}, $\bar{l}_{3,4}$, resonance contributions and scales μ are likewise generated. The outcoming one loop, two loops and total sum threshold parameters distributions provide the one loop, two loops and total sum central values and stantard deviations. For the latter, we use a 68% confidence level around the central value since the outcoming threshold parameters distributions are not gaussians, taking in this way into account possible skewness in the distributions. By using the Monte Carlo method we avoid summing errors in quadratures, which would be incorrect for statistically correlated quantities, and we do not have to use any complicated covariance formula.

In Ref. [4] the hardfull calculation of the two loop contributions to the threshold parameters was undertaken and explicit expressions for scattering lengths and ranges were written, in terms of the low energy constants, $b_{1,2,3,4,5,6}$. In this note we profit from their expressions and complete their numerical

[4]We assume a 10% error for this since resonance saturation is obviously an approximation. The total errors are never dominated by this assumption.

calculation by providing their numbers with the inherited error-bars as shown in Table 1. As we see, for parameter Set **I** the predictive power is lost for a_{20}, b_{20} and a_{22} with the present experimental accuracy, since the one loop errors are larger than the central values of the two loop contribution. The situation worsens dramatically, for parameter Set **II** where we see that in most considered cases predictive power is lost beyond one loop, with the exception of the S-wave scattering lengths a_{00} and a_{02}. From these results one concludes also, as expected, that K_{l4} is a more accurate method to predict the $\pi\pi$ low energy threshold parameters to two loops, yet more accurate determinations are needed. The results of the table are compatible, both for parameter sets **I** and **II** with the experimental analysis of $\pi\pi$ scattering data [6] but produce in general much better errors. Finally, the values of the b parameters with their errors turn out to be for Set **I** and **II** (in brackets),

$$\begin{aligned}
10 \cdot b_1 &= -0.92 \pm 0.21 \quad (-0.81 \pm 0.48) \\
10 \cdot b_2 &= +0.89 \pm 0.18 \quad (+0.77 \pm 0.42) \\
10^2 \cdot b_3 &= -0.47 \,{}^{+\,0.17}_{-\,0.12} \quad (-0.27 \pm 0.65) \\
10^2 \cdot b_4 &= +0.71 \pm 0.08 \quad (+0.48 \pm 0.13) \\
10^3 \cdot b_5 &= +0.25 \,{}^{+\,0.09}_{-\,0.14} \quad (+0.09 \,{}^{+\,0.24}_{-\,0.30}\,) \\
10^3 \cdot b_6 &= +0.14 \,{}^{+\,0.02}_{-\,0.04} \quad (+0.086 \,{}^{+\,0.013}_{-\,0.024}\,)
\end{aligned} \tag{7}$$

A few comments are in order. For the parameter Set **I**, the errors in the one loop parameters are more significant than the scale uncertainty quoted in ref. [4]. Notice that our central numbers for parameter Set **I** might be, roughly speaking, intermediate values between those quoted in Ref. [4], since they take the extreme values $\mu = 0.5$GeV and $\mu = 1$GeV, and we take $\mu = .75 \pm .25$GeV and the $\bar{l}_{1,2,3,4}$ are identical to theirs. We see that this is not always the case due to non-linearities in the scale dependence, and hence in this particular example one sees that this method of estimating the error by computing extreme values underestimates the error. For the parameter Set **II** the central values of the b parameters are slightly different than those in Ref. [4] because the central $\bar{l}_1$ and $\bar{l}_2$ values are also different. The errors for the parameter Set **II** are much larger than those found for parameter Set **I**. The size of the errors for Set **II** is comparable with the ones for b_3, b_4, b_5, b_6 found in Ref. [3]. The values of the parameters are compatible for both sets within error-bars, with the only exception of b_4 which is incompatible at the one standard deviation level.

In summary, effective field theories like Chiral Perturbation Theory have a limited predictive power because of three reasons: 1) truncation of the expansion, 2) proliferation of undetermined constants at any order and 3) experimental uncertainties. Thus, experimental data prove crucial to determine the, increasing with the order, unknown constants and their errors, which propagate in a correlated way to higher orders in the expansion possibly undermining the "convergence" of the expansion. This situation also appears in fundamental theories like QED or QCD, but it is in fact worse in ChPT because the number of unknown parameters in these theories does not increase with the order of the expansion. We have exemplified our points in the calculation of the threshold parameters for $\pi\pi$ scattering up to two loops. We find cases that, within the present experimental accuracy, predictive power is already lost at the two loop level, i.e., the errors in the one loop contribution are larger than the central values of the two loop contribution.

ChPT defines a whole family of effective theories, but obviously the most interesting choice exactly corresponds to low energy QCD. In QCD with two flavours and neglecting isospin breaking there appear only two independent parameters, $\Lambda_{\rm QCD}$ and the quark mass m_Q. The quantities f_π, m_π, the $\bar{l}$'s, the b's and higher order parameters must be functions of them. This dependence introduces correlations among all low energy parameters which, if taken into account, would influence the present error analysis, and perhaps might yield to more moderate errors in terms of $\Lambda_{\rm QCD}$ and m_Q and their errors. Assuming more statistically independent parameters than QCD suggests is, so far, another manifestation of the inability to undertake a quantitative and microscopic derivation of ChPT as an effective low energy theory of QCD on the one hand, but on the other hand reassures ChPT as a convenient tool to deal with non-perturbative phenomena in strong interactions. A way to make the effective field theory predictive for increasing orders in the chiral expansion is to perform more accurate measurements.

	Set	(tree)	+(1loop)	+(2loop)	total	experiment
a_{00}	**I**	0.156	0.044 ± 0.005	0.016 ± 0.003	0.216 ± 0.009	0.26 ± 0.05
a_{00}	**II**	0.156	0.039 ± 0.008	0.013 ± 0.003	0.208 ± 0.011	0.26 ± 0.05
b_{00}	**I**	0.179	0.069 ± 0.010	0.027 ± 0.007	0.275 ± 0.016	0.25 ± 0.03
b_{00}	**II**	0.179	0.059 ± 0.024	0.019 ± 0.011	0.256 ± 0.034	0.25 ± 0.03
$10 \cdot a_{11}$	**I**	0.297	0.073 ± 0.010	0.025 ± 0.006	0.395 ± 0.014	0.38 ± 0.02
$10 \cdot a_{11}$	**II**	0.297	0.058 ± 0.033	0.018 ± 0.005	0.374 ± 0.034	0.38 ± 0.02
$10 \cdot b_{11}$	**I**	0	0.048 ± 0.006	$0.031\ ^{+\ 0.005}_{-\ 0.007}$	$0.080\ ^{+\ 0.007}_{-\ 0.009}$	–
$10 \cdot b_{11}$	**II**	0	0.034 ± 0.033	$0.020\ ^{+\ 0.005}_{-\ 0.008}$	0.054 ± 0.029	–
$10 \cdot a_{20}$	**I**	-0.446	0.028 ± 0.018	0.004 ± 0.002	-0.414 ± 0.020	-0.28 ± 0.12
$10 \cdot a_{20}$	**II**	-0.446	0.008 ± 0.031	$0.000\ ^{+\ 0.002}_{-\ 0.003}$	-0.438 ± 0.032	-0.28 ± 0.12
$10 \cdot b_{20}$	**I**	-0.892	0.166 ± 0.038	$0.009\ ^{+\ 0.007}_{-\ 0.006}$	-0.717 ± 0.036	-0.82 ± 0.08
$10 \cdot b_{20}$	**II**	-0.892	0.098 ± 0.052	0.003 ± 0.011	-0.791 ± 0.046	-0.82 ± 0.08
$10^2 \cdot a_{02}$	**I**	0	0.181 ± 0.025	0.079 ± 0.016	0.260 ± 0.036	0.17 ± 0.03
$10^2 \cdot a_{02}$	**II**	0	0.117 ± 0.026	0.053 ± 0.018	0.170 ± 0.030	0.17 ± 0.03
$10^3 \cdot a_{22}$	**I**	0	0.21 ± 0.13	$-0.01\ ^{+\ 0.06}_{-\ 0.04}$	0.20 ± 0.10	0.13 ± 0.30
$10^3 \cdot a_{22}$	**II**	0	0.12 ± 0.44	$0.01\ ^{+\ 0.17}_{-\ 0.12}$	0.13 ± 0.30	0.13 ± 0.30

Table 1: Threshold $\pi\pi$ scattering parameters and their uncertainties in units of m_π due to the error bars in the $\bar{l}'s$ one loop parameters and the uncertainties in both the scale and the resonant part of the two loop contribution. Set **I** corresponds to K_{l4} one loop calculation: $\bar{l}_1 = -1.7 \pm 1.0\,, \bar{l}_2 = 6.1 \pm 0.5\,, \bar{l}_3 = 2.9 \pm 2.4\,, \bar{l}_4 = 4.3 \pm 0.9$. Set **II** corresponds to a two loop D-wave $\pi\pi$ scattering lengths calculation (see main text): $\bar{l}_1 = -0.8 \pm 4.8\,, \bar{l}_2 = 4.45 \pm 1.1\,, \bar{l}_3 = 2.9 \pm 2.4\,, \bar{l}_4 = 4.3 \pm 0.9$ The scale is $\mu = 750 \pm 250$MeV under the resonance saturation hypothesis, for which an error of 10% is assumed. Errors are *not* added in quadrature due to statistical correlations. Experimental values are from ref. [6].

References

[1] S. Weinberg, Phys. Rev. Lett **17** (1966) 616,

[2] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158** (1984) 142.

[3] M. Knecht, B. Moussallam, J. Stern and N.H. Fuchs, Nucl. Phys. **B457** (1995)513; *ibidem* **B471** (1996)445;

[4] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M.E. Sainio, Phys. Lett. **B374** (1996) 210; *ibidem* Nucl. Phys. **B508** (1997) 263.

[5] J. Bijnens, G. Colangelo and J. Gasser, Nucl. Phys. **B 427** (1994) 427.

[6] O. Dumbrajs et al., Nucl. Phys. **B 216** (1983) 277.